THE STABILITY OF THE UNPERTURBED MOTION OF A CERTAIN MECHANICAL SYSTEM

(OB USTAICHIVOSTI NEVOZMUSHCHENNOGO DVIZHENIIA ODNOI MEKHANICHESKOI SISTEMY)

PMM Vol.23, No.3, 1959, pp. 606-610

E.N. BEREZKIN
(Moscow)

(Received 29 January 1957)

In this paper the author considers a problem related to the stability of the unperturbed motion of an air plane with an autopilot. The characteristic equation of the first approximation of the system has a pair of zero roots with one group of solutions. General methods of solving this kind of problems were investigated by Liapunov [1] and Kamenkov [2]. In the actual cases the construction of the Liapunov function which determines the region of permissible perturbations may be very difficult.

1. We shall consider the system of equations of the perturbed motion

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = (b_1 - b_2 y) z^2, \qquad \frac{dz}{dt} = -ax - by - ez \tag{1.1}$$
$$(a > 0,\ b > 0,\ c > 0,\ b_1 > 0,\ b_2 > 0)$$

Through the linear substitution

$$x_1 = x, \qquad y_1 = y, \qquad z_1 = \alpha x + \beta y + \gamma z, \qquad \alpha = ac, \qquad \beta = bc - a, \qquad \gamma = c^2 \tag{1.2}$$

the system (1.1) is transformed into the canonical system

$$\begin{aligned} \frac{dx_1}{dt} &= y, \\ \frac{dy_1}{dt} &= a_1 x_1^2 + (a_2 x_1 + a_3 x_1^2) y_1 + (a_4 + a_5 x_1) y_1^2 + a_6 y_1^3 + a_7 z_1^2 + a_8 x_1 z_1 + \\ &\quad + (a_9 z_1 + a_{10} z_1^2 + a_{11} x_1 z_1) y_1 + a_{12} z_1 y_1^2 \\ \frac{dz_1}{dt} &= -cz_1 + m\,[a_1 x_1^2 + (a_2 x_1 + a_3 x_1^2) y_1 + (a_4 + a_5 x_1) y_1^2 + a_6 y_1^3 + a_7 z_1^2 + a_8 x_1 z_1 + \\ &\quad + (a_9 z_1 + a_{10} z_1^2 + a_{11} x_1 z_1) y_1 + a_{12} z_1 y_1^2] \end{aligned} \tag{1.3}$$

where a_i and m are known functions of the quantities a, b, c, b_1, b_2.

It is obvious that the problem of stability with respect to the new variables is equivalent to the problem of stability with respect to the old variables. We shall prove that in the case under consideration the unperturbed motion $x_1 = y_1 = z_1 = 0$ is unstable. For this purpose we shall perform a preliminary transformation of the system (1.3) by the substitution

$$x_1 = \xi, \qquad y_1 = \frac{\eta}{1 + Az_1} + B\xi z_1 + Dz_1^2$$

$$A = \frac{a_9 c + a_8}{c^2}, \qquad B = -\frac{a_8}{c}, \qquad D = -\frac{a_7}{2c} \tag{1.4}$$

In this case

$$\xi = x_1, \qquad \eta = (y_1 - Bx_1 z_1 - Dz_1^2)(1 + Az_1) \tag{1.5}$$

When $z = 0$ the substitution reduces to $x_1 = \xi$, $y_1 = \eta$. On the other hand, the terms of the substitution which contain z are all of the same degree, which is at least two. The above substitution could not lower the order of the right members of the differential equations. This can easily be proved by differentiating (1.5) with respect to the time t. We shall have

$$\frac{d\eta}{dt} = \frac{dy_1}{dt} - Bz_1\frac{dx_1}{dt} - Bx_1\frac{dz_1}{dt} + Az_1\frac{dy_1}{dt} + Ay_1\frac{dz_1}{dt} - 2Dz_1\frac{dz_1}{dt} - ABz_1^2\frac{dx_1}{dt} -$$
$$- 2ABx_1 z_1\frac{dz_1}{dt} - 3ADz_1^2\frac{dz_1}{dt} \tag{1.6}$$

Let us group together all the second order terms in the right-hand member of the above equation. With the old variables we shall have

$$a_1x_1^2 + a_2x_1y_1 + a_4y_1^2 + a_7z_1^2 + a_8x_1z_1 + a_9y_1z_1 - By_1z_1 + Bcx_1z_1 - Acy_1z_1 + 2Dcz_1^2. \tag{1.7}$$

The transformation from the old to the new variables ξ, η will leave the second order terms unchanged, but will introduce additional terms of a higher order. With A, B, and D defined as above we shall obtain the relation

$$(a_7 + 2Dc)z_1^2 + (a_8 + Bc)x_1z_1 + (a_9 - B - Ac)y_1z_1 \equiv 0 \tag{1.8}$$

The final form of the transformed system could be written as

$$\frac{d\xi}{dt} = \frac{\eta}{1 + Az_1} + B\xi z_1 + Dz_1^2$$
$$\frac{d\eta}{dt} = a_1\xi^2 + a_2\xi\eta + a_4\eta^2 + f_1(\xi, \eta, z_1) \tag{1.9}$$
$$\frac{dz_1}{dt} = -cz_1 + f_2(\xi, \eta, z_1)$$

Here the function $f_1(\xi, \eta, z_1)$ contains no terms lower than the third order with respect to the variables ξ, η, z_1; the function $f_2(\xi, \eta, z_1)$ contains no terms lower than the second order with respect to these variables.

Let us consider the function

$$V(\xi, \eta, z_1) = \eta + (a_1 - a_4)\xi\eta - \frac{1}{2}a_2\xi^2 - \frac{a_1}{2c}z_1^2 \tag{1.10}$$

In any neighborhood of the unperturbed motion $\xi = \eta = z_1 = 0$ there are points where $V(\xi, \eta, z_1) > 0$, and also points where $V(\xi, \eta, z_1) < 0$. The time derivative of the function V equals

$$\frac{dV}{dt} = [(a_1 - a_4)\,\eta - a_2\xi]\left[\frac{\eta}{1 + Az_1} + B\xi z_1 + Dz_1\right] + [1 + (a_1 - a_4)\,\xi]\,[a_1\xi^2 + a_2\xi\eta + a_4\eta^2 + \\ + f_1(\xi, \eta, z_1)] + a_1 z_1^2 - \frac{a_1}{c} z_1 f_1(\xi, \eta, z_1) \equiv a_1(\xi^2 + \eta^2 + z_1^2) + \Phi(\xi, \eta, z_1) \qquad (1.11)$$

Here $\Phi(\xi, \eta, z_1)$ contains no terms lower then of the third order of magnitude.

Thus the function V satisfies all the conditions of Liapunov's theorem on the instability of motion [3]. The conditions of instability would not be satisfied if $a_1 = 0$, which is possible when $a = 0$, or when $b_1 = 0$.

2. We shall consider now the case when $a \equiv 0$, $b_1 \neq 0$. In this case equations (1.3) will become

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = a_4 y^2 + a_6 y^3 + a_7 z^2 + a_9 yz + a_{10} yz^2 + a_{12} y^2 z \qquad (2.1)$$
$$\frac{dz}{dt} = -cz + m\,[a_4 y^2 + a_6 y^3 + a_7 z^2 + a_9 yz + a_{10} yz^2 + a_{12} y^2 z]$$

We shall show that in this case the unperturbed motion is unstable. In order to achieve this we shall consider the function of Chetaev

$$V(x, y, z) = xy - \frac{z^2}{2c} \qquad (2.2)$$

The region $V > 0$ is represented by the inside surface of a cone. Taking into account the system (2.1), the derivative of V is

$$\frac{dV}{dt} = y^2\left[1 + a_4 x + a_6 xy + a_{12} xz - \frac{ma_4}{c} z - \frac{ma_6}{c} yz - \frac{ma_{12}}{c} z^2\right] + \\ + z^2\left[1 + a_7 x + a_{10} xy - \frac{ma_7}{c} z - \frac{ma_9}{c} y - \frac{ma_{10}}{c} yz\right] + a_9 xyz \qquad (2.3)$$

In a sufficiently small neighborhood of the unperturbed motion the sign or the right member of this equation is determined by the sign of the expression

$$f(x, y, z) = y^2 + z^2 + a_9 xyz \qquad (2.4)$$

The above expression is a quadratic form with respect to the variables y and z, and is a positive-definite function for all values of x satisfying the condition

$$|x| < \left|\frac{2}{a_9}\right| = \frac{c^3}{bb_1} \qquad (2.5)$$

For given values of x the function $f(xyz)$ could vanish only on the line $y = z = 0$ lying on the surface of the cone:

$$xy - \frac{z^2}{2c} = 0 \qquad (2.6)$$

Thus, the constructed function V satisfies the conditions of Chetaev's theorem on the instability of motion [4], which proves our statement. It is easy to notice that the function V would also satisfy the conditions of Chetaev's theorem when $a = b_1 = 0$. In this last case the right member of the inequality (2.3) becomes

$$y^2\left[1 + a_6 xy + a_{12}xz - \frac{ma_6}{c} yz - \frac{ma_{12}}{c} z^2\right] + z^2\left[1 + a_{10}xy - \frac{ma_{10}}{c} yz\right] \qquad (2.7)$$

and represents a positive-definite function for any sufficiently small value of the variables x, y, z, which vanishes in the neighborhood of the unperturbed motion on the surface of the cone (2.6). Thus, in this last case the unperturbed motion $x = y = z = 0$ is also unstable.

3. We shall consider now the case when $b_1 = 0$, $a \neq 0$. The equations (1.3) become in this case

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = a_1x^2y + a_2xy^2 + a_3y^3 + a_4yz^2 + a_5xyz + a_6y^2z \qquad (3.1)$$

$$\frac{dz}{dt} = -cz + m\left[a_1x^2y + a_2xy^2 + a_3y^3 + a_4yz^2 + a_5xyz + a_6y^2z\right]$$

where the coefficients a_i, m will depend only on the given values a, b, c, b_1, b_2.

The direct construction of Liapunov's function for the system (3.1) presents considerable difficulties, which could be circumvented through an additional transformation of the system achieved by the following substitutions:

$$x = x_1, \quad y = y_1, \quad z = z_1 + \frac{ma_1x_1^2}{c + a_1x^2} y_1 + \frac{m}{c}\left(a_2 - 2\frac{a_1}{c}\right)x_1y_1^2 + \frac{m}{c}\left(a_3 - \frac{a_2}{c} + 2\frac{a_1}{c^2}\right)y_1^3 \equiv z_1 + U(x_1y_1) \qquad (3.2)$$

$$x_1 = x, \quad y_1 = y, \quad z_1 = z - \frac{ma_1x^2}{c + a_1x^2} y - \frac{m}{c}\left(a_2 - 2\frac{a_1}{c}\right)xy^2 - \frac{m}{c}\left(a_3 - \frac{a_2}{c} + 2\frac{a_1}{c^2}\right)y^3 \equiv z - U(xy)$$

$$x_1 = x_2, \quad y_1 = \frac{-1 + \sqrt{1 + 4\alpha y_2z_2}}{2\alpha z_2}, \quad z_1 = z_2, \quad \alpha = \frac{a_6}{c} \qquad (3.3)$$

$$x_2 = x_1, \; y_2 = y_1 + \alpha y_1^2 z, \quad z_2 = z_1$$

The substitution of (3.2) into (3.1) will change the system (3.1) as follows: the first equation will not change; in the second equation will appear additional terms of fifth order; the third equation is the only one which will show some essential changes.

On the strength of the identity

$$ma_1x_1^2y_1 + ma_2x_1y_1^2 + ma_3y_1^3 - \frac{2ma_1c}{(c + a_1x_1^2)^2}xy^2 - \frac{m}{c}\left(a_2 - 2\frac{a_1}{c}\right)y_1^3 - \frac{ma_1^2x_1^4y_1}{c + a_1x_1^2} -$$
$$-\frac{ma_1cx_1^2y_1}{c + a_1x_1^2} - m\left(a_2 - 2\frac{a_1}{c}\right)x_1y_1^2 - m\left(a_3 - \frac{a_2}{c} + 2\frac{a_1}{c^2}\right)y_1^3 = 2ma_1^2\frac{2c + a_1x_1^2}{c(c + a_1x_1^2)^2}x^3y^2 \quad (3.4)$$

the system (3.1) transforms into

$$\frac{dx_1}{dt} = y_1$$
$$\frac{dy_1}{dt} = a_1x_1^2y_1 + a_2x_1y_1^2 + a_3y_1^3 + a_5x_1y_1z_1 + a_6z_1y_1^2 + f(x_1y_1z_1) \quad (3.5)$$
$$\frac{dz_1}{dt} = -cz_1 + f_2(x_1y_1z_1)$$

and the functions $f_1(x_1y_1z_1)$ and $f_2(x_1y_1z_1)$ could be expressed as

$$f_1(x_1y_1z_1) = z_1^2\varphi_1(x_1y_1z_1) + x_1^2y_1^2\varphi_2(x_1y_1z_1) + x_1y_1^3\varphi_3(x_1y_1z_1) + y_1^4\varphi_4(x_1y_1z_1)$$
$$f_2(x_1y_1z_1) = z_1\varphi_5(x_1y_1z_1) + x_1^2y_1^2\varphi_6(x_1y_1z_1) + y_1^4\varphi_7(x_1y_1z_1) \quad (3.6)$$
$$\varphi_k(0, 0, 0) = 0 \qquad (k = 1, \ldots, 7)$$

It is easy to notice that the substitution (3.2) does not lower the order of terms in the right members in the system (3.5). After the substitution (3.3) the system becomes

$$\frac{dx_2}{dt} = \frac{-1 + \sqrt{1 + 4\alpha y_2z_2}}{2\alpha z_2} = y_2 - \frac{a_6}{c}y_2^2z_2 + 2\frac{a_6^2}{c^2}y_2^3z_2^2 + \cdots, \alpha = \frac{a_6}{c}$$
$$\frac{dy_2}{dt} = a_1x_1^2y_1 + a_2x_1y_1^2 + a_3y_1^3 + a_5x_1y_1z_1 + f_1(x_1y_1z_1) + 2\frac{a_6}{c}[a_1x_1y_1^2z_1 + a_2x_1y_1^3z_1 +$$
$$+ a_3y_1^4z_1 + a_5x_1y_1^2z_1^2 + a_6z_1^2y_1^3 + y_1z_1f_1(x, y, z)] + \frac{a_6}{c}y_1^2f_2(x_1y_1z_1)$$
$$\frac{dz_2}{dt} = -cz_1 + f_2(x_1y_1z_1) \quad (3.7)$$

Finally, after dropping the indices, the transformed system could be written in the form:

$$\frac{dx}{dt} = y + \Phi_1(yz)$$
$$\frac{dy}{dt} = a_1x^2y + a_2xy^2 + a_3y^3 + a_5xyz + \Phi_2(xyz) \quad (3.8)$$
$$\frac{dz}{dt} = -cz + \Phi_3(xyz)$$

where

$$\Phi_1(y, z) = \frac{-1 + \sqrt{1 + 4\alpha yz}}{2\alpha z} - y = -\frac{a^6}{c}zy^2 + 2\frac{a_6^2}{c^2}z^2y^3 + \ldots$$
$$\Phi_2(xyz) = z^2F_1(xyz) + x^2y^2F_2(xyz) + y^4F_3(xyz) + xy^3F_4(xyz) \quad (3.9)$$
$$\Phi_3(xyz) = zF_5(xyz) + x^2y^2F_6(xyz) + y^4F_7(xyz).$$
$$F_k(0, 0, 0) = 0 \qquad (k = 1, \ldots, 7).$$

Let us consider Liapunov's function

$$V(xyz) = \left\{ y\left[\exp\left(-\frac{a_2x^2}{2}\right)\right] - \int_0^x a_1x^2\left[\exp\left(-\frac{a_2x^2}{2}\right)\right]dx - a_3y^2\int_0^x\left[\exp\left(-\frac{a_2x^2}{2}\right)\right]dx\right\}^2 + \\ + y^2[\exp(-a_2x^2)] - y^3\left[\exp\left(-\frac{3}{2}a_3x^2\right)\right]\int_0^x(1+2a_3[\exp(-a_2x^2)])\left[\exp\left(\frac{3}{2}a_3x^2\right)\right]dx + \frac{z^2}{2c} \tag{3.10}$$

In the neighborhood of the unperturbed motion for sufficiently small values of the variables x and y the sign of the above function is determined by the sign of the polynomial.

$$\left(y - \frac{a_1x^3}{3} - a_3xy^2\right)^2 + y^2 + \frac{z^2}{2c} \tag{3.11}$$

which can vanish only when

$$x = y = z = 0. \tag{3.12}$$

It follows that the function $V(x, y, z)$ is a positive-definite function of the variables x, y, z. Taking into account the system (3.8), the total time derivative of the function V equals

$$\frac{dV}{dt} = y^2\left\{2a_1x^2\exp(-a_2x^2) + 4a_1a_3x_2\int_0^x\left[\exp\left(-\frac{a_2x^2}{2}\right)\right]dx\int_0^x a_1x^2\left[\exp\left(-\frac{a_2x^2}{2}\right)\right]dx\right\} + \\ + y^3\left[4a_1a_3x^2\exp\left(-\frac{a_2x^2}{2}\right)\int_0^x\exp\left(-\frac{a_2x^2}{2}\right)dx - 3a_1x^2\exp\left(-\frac{3}{2}a_3x^2\right)\times\right. \\ \times\int_0^x[1+2a_3\exp(-a_2x^2)]\exp\left(\frac{3}{2}a_3x^2\right)dx + 4a_2a_3x\int_0^x\exp\left(-\frac{a_2x^2}{2}\right)dx\int_0^x a_1x^2\times \\ \left.\times\text{epx}\left(-\frac{a_2x^2}{2}\right)dx\right] + y^4\left[-1 - 4a_2a_3x\exp\left(-\frac{a_2x^2}{2}\right)\int_0^x\exp\left(-\frac{a_2x^2}{2}\right)dx + \right. \\ + 3(a_3 - a_3)x\exp\left(-\frac{3}{2}a_3x^2\right)\int_0^x[1+2a_3\exp(-a_2x^2)]\exp\left(\frac{3}{2}a_3x^2\right)dx + \\ \left. + 4a_1a_3^2\left(\int_0^x\exp\left(-\frac{ax^2}{2}\right)dx\right)^2 + 4a_3^2\int_0^x\exp\left(-\frac{a_2x^2}{2}\right)dx\int_0^x a_1x^2\exp\left(-\frac{a_2x^2}{2}\right)dx\right] + \\ + y^5\left[-4a_3^2\exp\left(-\frac{a_2x^2}{2}\right)\int_0^x\exp\left(-\frac{a_2x^2}{2}\right)dx + 4a_2a_3^2x\left(\int_0^x\exp\left(-\frac{a_2x^2}{2}\right)dx\right)^2 - \right. \\ \left. - 3a_3\exp\left(-\frac{3}{2}a_3x^2\right)\int_0^x[1+2a_3\exp(-a_2x^2)]\exp\left(\frac{3}{2}a_3x^2\right)dx\right] +$$

$$+ y^6 4a_3^3 \left(\int_0^x \exp\left(-\frac{a_2x^2}{2}\right)dx\right)^2 - z^2 + \frac{\partial V}{\partial x}\Phi_1(x, y, z) + \frac{\partial V}{\partial y}\Phi_2(x, y, z) +$$
$$+ \frac{z}{c}\Phi_3(x, y, z) + \frac{\partial V}{\partial y}a_5xyz \qquad (3.13)$$

For sufficiently small values of x and y the above derivative could be written as

$$\frac{dV}{dt} = 2a_1x^2y^2[1 + \gamma_1(xyz)] - y^4[1 + \gamma_2(xyz)] - z^2[1 + \gamma_3(xyz)] + \frac{\partial V}{\partial y}a_5xyz \qquad (3.14)$$

Here, $\gamma_1(x, y, z)$, $\gamma_2(x, y, z)$, $\gamma_3(x, y, z)$ are continuous functions of x, y, z, in the neighborhood of the unperturbed motion, and vanishing at the point $x = y = z = 0$. In a certain neighborhood S_1 of the unperturbed motion, where the following conditions are satisfied:

$$|\gamma_1(x,y,z)| < \delta, \quad |\gamma_2(x,y,z)| < \delta, \quad |\gamma_3(x,y,z)| < \delta, \quad \delta = \text{const} < 1 \qquad (3.15)$$

we shall have

$$-2a_1x^2y^2[1 + \gamma_1(x,y,z)] + y^4[1 + \gamma_2(x,y,z)] + z^2[1 + \gamma_3(x,y,z)] - \frac{\partial V}{\partial y}a_5xyz \geqslant$$
$$\geqslant (1 - \delta)[-2a_1x^2y^2 + y^4 + z^2] - \frac{\partial V}{\partial y}a_5xyz \geqslant (1 - \delta)[-2a_1x^2y^2 + y^4 + z^2 - k|xyz|] \qquad (3.16)$$

where k represents the maximum value of $|a_5\partial V/\partial y|$ in a certain neighborhood S_2 of the unperturbed motion. The function

$$f(x,y,z) = -2a_1x^2y^2 + z^2 - k|xyz| \qquad (3.17)$$

is a positive-definite quadratic form with respect to the variables $u = xy$, z for all values of k satisfying the condition

$$k^2 < -8a_1 \qquad (3.18)$$

The function $f(xyz)$ vanishes at $x = z = 0$, $y \neq 0$, and at $y = z = 0$, $x \neq 0$, and hence is a positive-definite function in the sense of Liapunov. We shall select now S_2 such that the condition (3.18) for k will be satisfied. Suppose there exist a neighborhood of the unperturbed motion (S) which is entirely in S_1 and S_2 as well. Then, in the neighborhood (S) the derivative dV/dt will be a negative-definite function of the variables x, y, z, vanishing on the line $y = z = 0$. In this way we have shown that the constructed function V satisfies all the conditions of Liapunov's theorem on the stability of the unperturbed motion.

It could be shown that the asymptotic stability does not hold in the considered case. This is seen from the following solution of the system (3.8)

$$x \equiv x_0 = \text{const}, \qquad y \equiv z \equiv 0 \qquad (3.19)$$

BIBLIOGRAPHY

1. Liapunov, A.M., Issledovanie odnogo iz osobennykh sluchaev zadachi ob ustoichivosti (Investigation of a certain special case of the problem of stability of motion). *Mat. Sborn.* Vol. 17, No. 2, pp.253-333, 1883.

2. Kamenkov, G.V., Ob ustoichivosti dvizhenia (On the stability of motion). *Trud. Kazan. Aviats. Inst.* No. 9, 1939.

3. Liapunov, A.M., *Obshchaya zadacha ob ustoichivosti dvizhenia (The general problem of the stability of motion)*. Gostekhteoretizdat, 1950.

4. Chetaev, N.G., *Ustoichivost dvizhenia (Stability of motion)*. Gostekhizdat, 1955.

Translated by T.L.